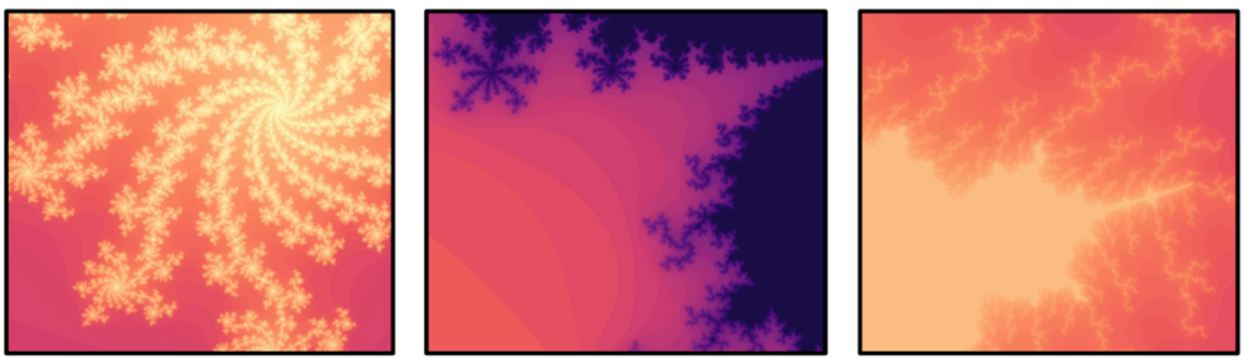
Investigating the Mandelbrot Set and Estimating Pi

Advik Chaudhary



1 Introduction

Motivation: I remember being fascinated by depictions of the Mandelbrot set on the covers of maths textbooks and in articles. Additionally, I learnt that the Set exhibited unexpected patterns—such as leading to approximations of π—and I wished to prove such phenomena first-hand and understand the set in detail using knowledge from the course.

Specifically, I wanted to derive the escape condition (an important part of creating the set) and also prove how π originates from the Set. I will also be modifying the iterative equation used to create the set to understand how it affects the shape of the Set and whether the approximation of π in this variation still holds.

Structure: overall, the investigation is focused on deriving π from the Set—which first requires understanding the iterative sequence, then extracting π from the sequence, and finally whether other forms of the Set still exhibit this property.

2 Creating the Mandelbrot Set

2.1 Background: The $\mathcal{M}$ Sequence

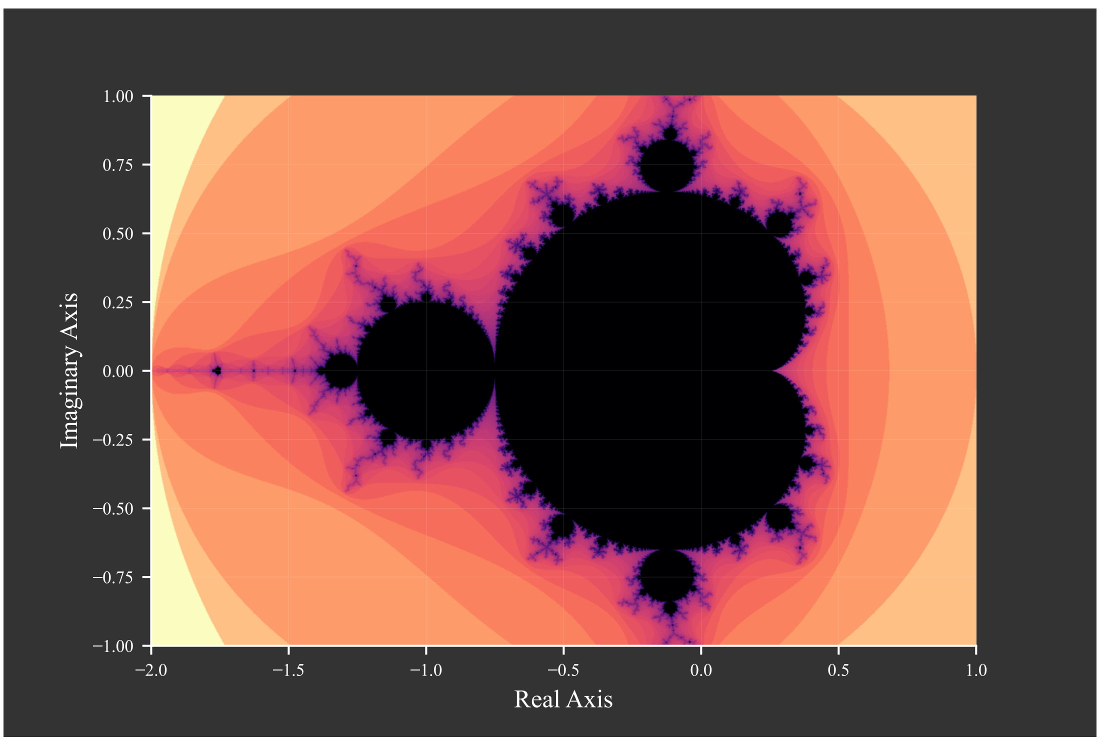


Figure 1: Image of the Mandelbrot set; I created this in Python with the `Matplotlib` graphing package.

To understand where π originates form, I first had understand and analyse the set in detail. Though it may seem complex, the Mandelbrot Set is created by a relatively simple equation and set of rules. The following equation creates the set:

$$z_{n+1} = z_n^2 + c \tag{2.1}$$

It is not a standard equation but a recursive one. It essentially states that to find some next term, z_{n+1}, we have to take the previous term, z_n, square it, and add a constant, c (all terms are complex numbers). Then, to calculate the third term, z_{n+2}, this process can be repeated with z_{n+1} now as the previous term. The initial term, z_0, is usually set to 0. Using this, we can generate the following sequence:

$$\begin{aligned}
z_0 &= 0 \\
z_1 &= z_0^2 + c = 0^2 + c = c \\
z_2 &= z_1^2 + c = c^2 + c \\
z_3 &= z_2^2 + c = (c^2 + c)^2 + c \\
&\vdots
\end{aligned} \tag{2.2}$$

This sequence will be referred to as the Mandelbrot or $\mathcal{M}$ sequence. The steps to create the Set using this sequence are:

1. Take a point on the complex plane.
2. Set this point as z_1 in the $\mathcal{M}$ sequence
3. Check if the magnitude of this term is greater than 2
4. If the magnitude is greater than 2, colour it
5. Find the next term in the $\mathcal{M}$ sequence based on the current term
6. Repeat steps 3-5 until the max limit of terms (set manually) has been reached or the magnitude of a term is greater than 2

The escape condition, checking whether each term's magnitude is greater than two, indicates whether the term's value in the sequence will eventually diverge to infinity (terms' values keep increasing) or converge to a particular value (terms' values stay stable and do not increase to infinity).

This process can be simplified to:

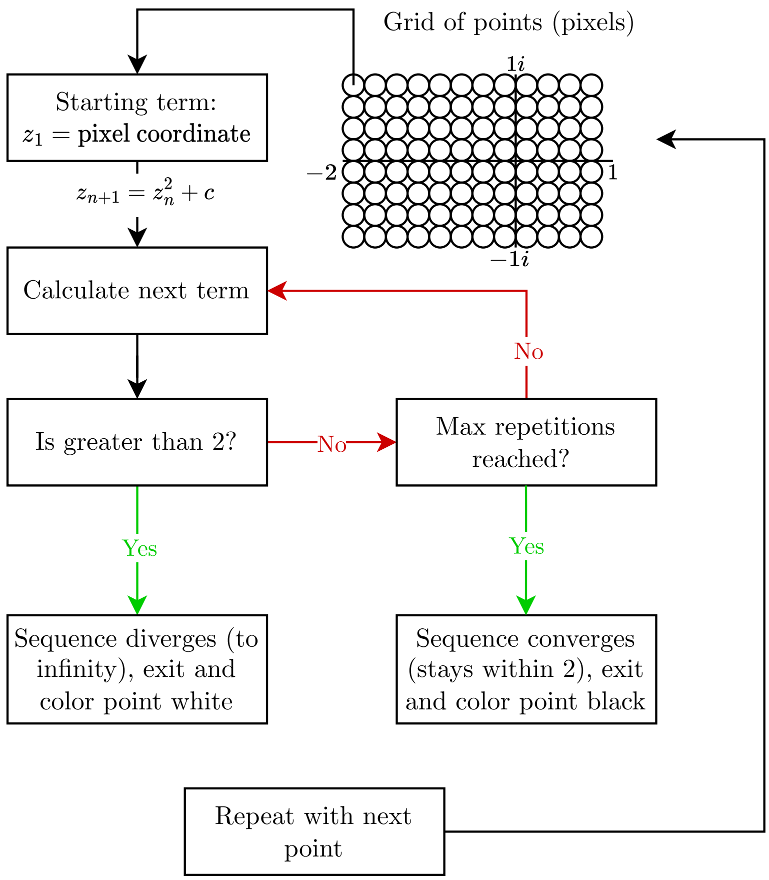


Figure 2: Flowchart of the steps to create the Mandelbrot set using an iterating function.

These steps color one point in the set. To form an image, a grid of points on the complex plane is used, each run through this procedure and coloured according to the results.

Thus, the Set simply shows whether, for a given point, will the $\mathcal{M}$ sequence diverge to infinity or not. (Note that creating a gradient of colours, instead of only two colours, comes from mapping the number of terms for $\mathcal{M}$ to diverge with a colour scheme.)

2.2 Classifying Starting Values

As stated previously, to determine whether a point lies within the set or not, the escape condition of $|z_n| > 2$ was used (whether a term's magnitude is greater than 2). I wanted to derive why this seemingly arbitrary condition is used with basic concepts including the triangle inequality and proof by induction.

I started by considering some term, z_a, in the sequence that was found to have a magnitude greater than 2 (not necessarily the first term):

$$|z_a| > 2$$

The next term, z_{a+1}, would then be

$$\begin{aligned} z_{a+1} &= z_a^2 + c \\ \implies |z_{a+1}| &= |z_a^2 + c| \end{aligned} \tag{2.3}$$

using the formula for the $\mathcal{M}$ sequence in (2.1).

I thought that imposing constraints on these variables would help prove this escape condition—I related the magnitutdes in (2.3) through the triangle inequality. Looking at Figure 3 below, we can see that $z_a^2 + c$ is formed by taking the vector z_a^2 and adding the c term to it, forming a triangle with sides of length $|z^2 + c|$, $|z^2|$, and $|c|$.

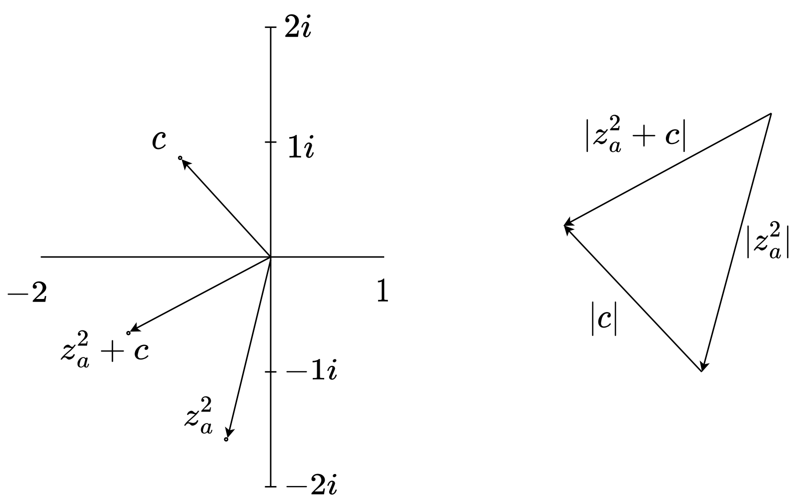


Figure 3: The triangle formed with z_a, the current term, c, and the next term, $z_{n+1} = |z_a^2 + c|$

Thus, by the triangle inequality (sum of any two sides must be larger than the third),

$$\begin{aligned} |z_a^2 + c| + |c| &\geq |z_a^2| \\ \implies |z_a^2 + c| &\geq |z_a^2| - |c| \\ \implies |z_{a+1}| &\geq |z_a^2| - |c| \end{aligned} \tag{2.4}$$

Assume $|z_a| \geq |c|$ (essentially that the current term greater than or equal to the first term). This means that,

$$-|z_a| \leq -|c| \tag{2.5}$$

Hence, we can safely replace $-|c|$ in the inequality (2.4) with $-|z_a|$:

$$\begin{aligned} &|z_{a+1}| \geq |z_a^2| - |z_a| \\ \implies &|z_{a+1}| \geq |z_a|(|z_a| - 1) \end{aligned} \tag{2.6}$$

Thus, we have a relation between the previous term and the next term in (2.6), through which we can understand how the sequence progresses. The relationship tells us that the next term is greater than the previous term, $|z_a|$, scaled by a factor of $|z_a| - 1$. Additionally, this factor is greater than 1 since $|z_a| > 2 \implies |z_a| - 1 > 1$. Thus, we can say the next term is strictly greater than the previous:

$$|z_{a+1}| = |z_a^2 + c| > |z_a|(|z_a| - 1) \tag{2.7}$$

From (2.7), I conjectured that the pattern in the sequence was $|z_{a+n}| > |z_a|(|z_a| - 1)^n$, essentially that a term n terms ahead of the current term, z_a, is exponentially greater in magnitude. Proving this would show that if we know the magnitude is greater than 2, then the terms will keep exponentially increasing to infinity.

2.3 Proving the Escape Condition by Induction

I thought I could use induction to prove that the conjecture, proven in the previous section, does not only hold for a term one step ahead of it but n steps ahead. In the following, $|z_{a+n}|$ is the term n steps ahead and $|z_a|$ is the current term.

Conjecture: $|z_{a+n}| > |z_a|(|z_a| - 1)^n$ for $|z_a| > 2$

Consider $n = 1$,

$|z_{a+1}| > |z_a|(|z_a| - 1)^1$ From (2.7)

Assume true for some $n = k$ where $k \in \mathbb{Z}^+$,

$|z_{a+k}| > |z_a|(|z_a| - 1)^k$

Consider $n = k + 1$,

$|z_{a+k+1}| = |z_k^2 + c|$

From (2.6),

$\geq |z_{a+k}|(|z_{a+k}| - 1)$

From assumption,

$= |z_a|(|z_a| - 1)^k \left[|z_a|(|z_a| - 1)^k - 1 \right]$

[1]Since $|z_a|(|z_a| - 1)^k >> |z_a| - 1$,

$> |z_a|(|z_a| - 1)^k(|z_a| - 1)$

$> |z_a|(|z_a| - 1)^{k+1}$

Thus, conjecture true for $n = k \implies$ true for $n = k + 1$

and since true for $n = 1$, true $\forall n \in \mathbb{Z}^+$

$$\therefore |z_{a+n}| > |z_a|(|z_a| - 1)^n \tag{2.8}$$

Hence, the escape condition has been proven. Since $|z_a| > 2$, $|z_a| - 1 > 1$. This means

[1]This is true as the factor $(|z_a| - 1)$ is more than 1 and increases $|z_a|$ on the LHS while the RHS subtracts one from $|z_a|$

that when this factor of $|z_a| - 1$ is raised to the nth power, it will approach ∞ as $n \to \infty$. Since this factor is multiplied by the previous term, $|z_a|$, in (2.8) to get the next nth term, $|z_{a+n}| \to \infty$.

This is where the '2' comes from—it is the smallest number for which we can say for sure that the sequence will diverge as the next term will be greater by a factor of more than 1 ($|z_a| - 1 > 1$).

Thus, $|z_a| > 2 \implies |z_{a+n}| \to \infty$ and the sequence diverges.

3 Approximating π with the Mandelbrot Set

3.1 The π Result in the Set

The Set exhibits an interesting pattern that I noticed when tweaking the starting values, originally found by D. Boll. When the starting term, z_1, is set closer and closer to the boundary of the Mandelbrot Set, the number of terms after which the sequence diverges (when $|z_n| > 2$) starts to approximate π.

For instance, take the point $0.25 + 0i$ on the boundary of the Set:

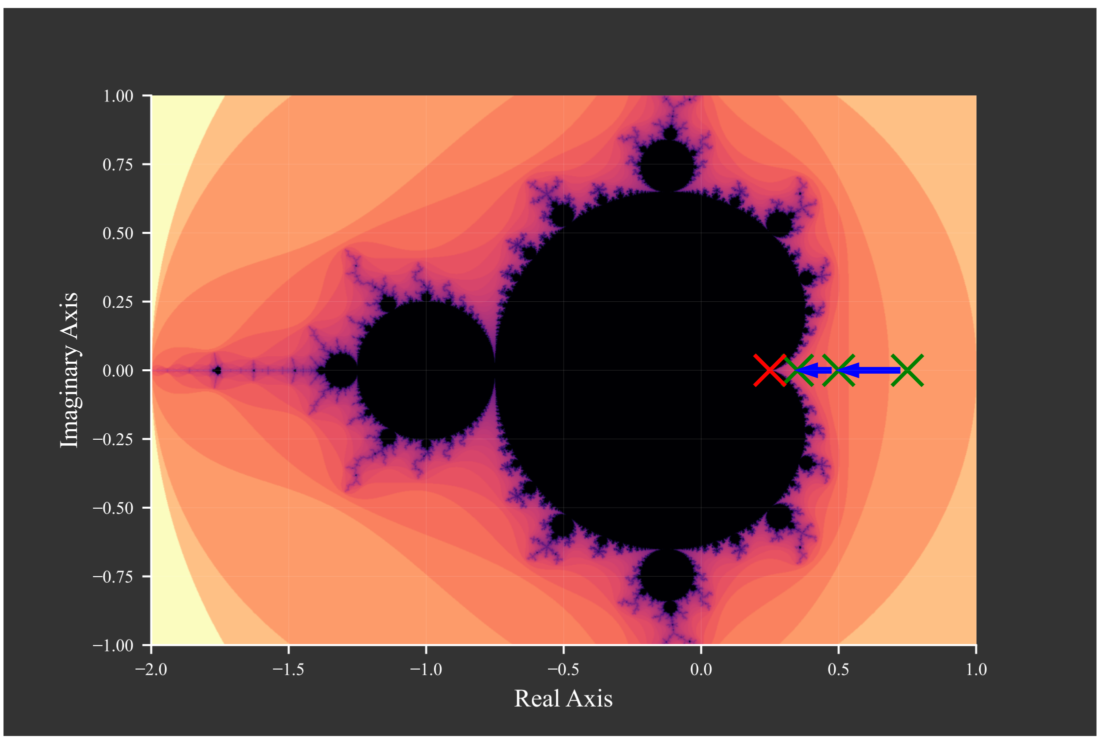


Figure 4: Approaching the boundary of the Mandelbrot set from 0.25^+ (red). Points for z_1 (green) are taken closer and closer to the right edge of the set.

As we take terms closer to $0.25 + 0i$, we get the following pattern:

a	$z_1 = 0.25 + a$	n_{div}
1.0	1.25	2
0.01	0.26	30
0.0001	0.2501	312
0.000001	0.250001	3140
0.00000001	0.25000001	31414
0.0000000001	0.2500000001	314157
0.000000000001	0.250000000001	3141591

Table 1: The π result as points closer to 0.25^+ are used for the initial value

Where a is the value added to 0.25 (essentially how much farther the point is from the edge of the set) to make the first term, z_1, and n_{div} is the number of terms after which this sequence diverges. Here, we see n_{div} approach π in digits.

3.2 Proving the π Result

I found finding π—or at least an approximation of it—in the Set intriguing and wanted to understand where this result comes from. I started with generating the $\mathcal{M}$ sequence, with starting term $0.25 + a$ specifically:

$$\begin{aligned} z_1 &= c = 0.25 + a \\ z_2 &= z_1^2 + 0.25 + a \\ z_3 &= z_2^2 + 0.25 + a \\ &\vdots \\ z_{n+1} &= (z_n)^2 + 0.25 + a \end{aligned} \tag{3.1}$$

To get the π result, it likely required some form of integration to somehow introduce π into the sequence. Thus, I converted (3.1) to a differential equation.

First, I created a $z_{n+1} - z_n$ term (which we can later write as dz) by completing the square:

$$\begin{aligned} z_{n+1} &= (z_n)^2 + \frac{1}{4} + a \\ z_{n+1} &= \left(z_n - \frac{1}{2}\right)^2 + z_n + a \\ z_{n+1} - z_n &= \left(z_n - \frac{1}{2}\right)^2 + a \end{aligned} \tag{3.2}$$

Since (3.2) tells us how the difference in value between two terms, dz, changes for a difference in term number, dn, we can convert this into a differential:

$$\frac{dz}{dn} = \left(z - \frac{1}{2}\right)^2 + a \tag{3.3}$$

Where z would give the value of the $\mathcal{M}$ sequence at the nth step. Forming an integral,

$$\int \frac{dz}{(z-\frac{1}{2})^2 + a} = \int dn$$

I recognized this as being in the format of the derivative of arctan since $\frac{d}{dx}\arctan(x) = \frac{1}{x^2+1}$. Thus, I tried to use the u-substitution method, replacing the contents of the squared value with u:

$$\int \frac{dz}{(\frac{2z-1}{2})^2 + a} = \int dn$$

$$\int \frac{dz}{a\left[\frac{1}{a}(\frac{2z-1}{2})^2 + 1\right]} = \int dn$$

Merging this $\frac{1}{a}$ value inside the square so that it can be substituted for with u,

$$\frac{1}{a}\int \frac{dz}{\left[(\frac{2z-1}{2\sqrt{a}})^2 + 1\right]} = \int dn$$

Now, we can use the substitution, $u = \frac{2z-1}{2\sqrt{a}}$ and $du = \frac{1}{\sqrt{a}}dz$,

$$\frac{1}{a}\int \frac{\sqrt{a}du}{u^2+1} = \int dn$$

$$\frac{1}{\sqrt{a}}\int \frac{du}{u^2+1} = \int dn$$

Integrating this with arctan,

$$\frac{\arctan(u)}{\sqrt{a}} + C = n$$

$$n = \frac{\arctan(\frac{2z-1}{2\sqrt{a}})}{\sqrt{a}} + C \tag{3.4}$$

Finally, solving for the constant of integration, C, using the condition $z = 0$ when $n = 0$ (since the $\mathcal{M}$ sequence has initial term $z_0 = 0$), I found:

$$C = -\frac{\arctan\left(\frac{-1}{2\sqrt{a}}\right)}{\sqrt{a}} \tag{3.5}$$

Substituting this constant into (3.4) and rearranging for z,

$$z = \sqrt{a}\tan\left(n\sqrt{a} + \arctan\left(\frac{-1}{2\sqrt{a}}\right)\right) + \frac{1}{2} \tag{3.6}$$

Thus, we have an equation for the nth term, z, of the $\mathcal{M}$ sequence in terms of term's number, n. We can plot this using an a of 0.01—dictating how far the starting point is from the edge of the set—with the value of the nth term on they y-axis and the term count on the x-axis:

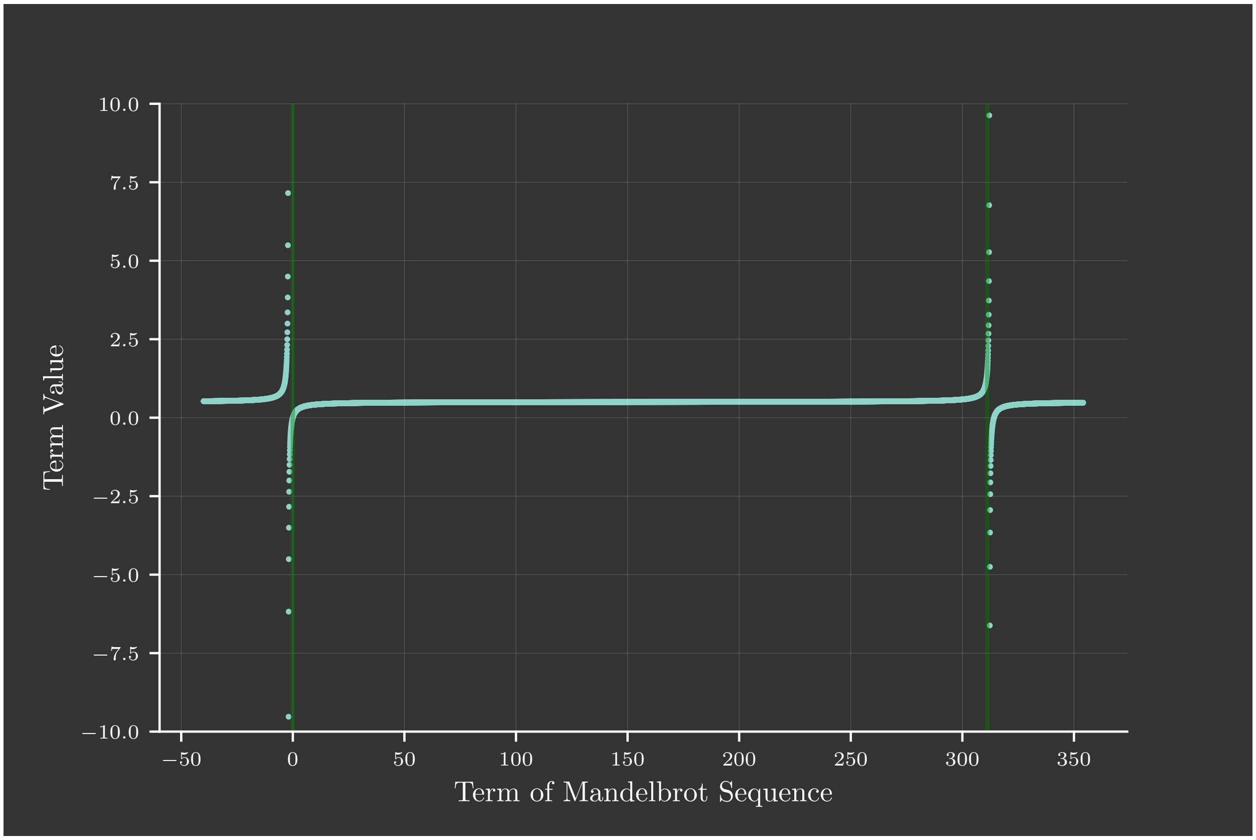


Figure 5: The Mandelbrot sequence plotted for starting term 0.25 + a for a = 0.0001. The sequence starting from the initial term to the point where it diverges is shown by the green bounds.

Since the equation for the nth term includes a tan function, we get repeating, asymptotic behaviour. However, this graph is only valid from 0 to the point at which the term value tends to infinity (the "middle section" of Figure. 5 within the green bounds). Specifically, this is the range $0 \leq n \leq \frac{\pi}{\sqrt{a}}$.

Within this valid range, we can see the term numbers are positive and the terms are increasing in value, diverging to infinity—which indicates this point is not in the Set. Zooming into the valid tan part, we can see how the sequence progresses more clearly:

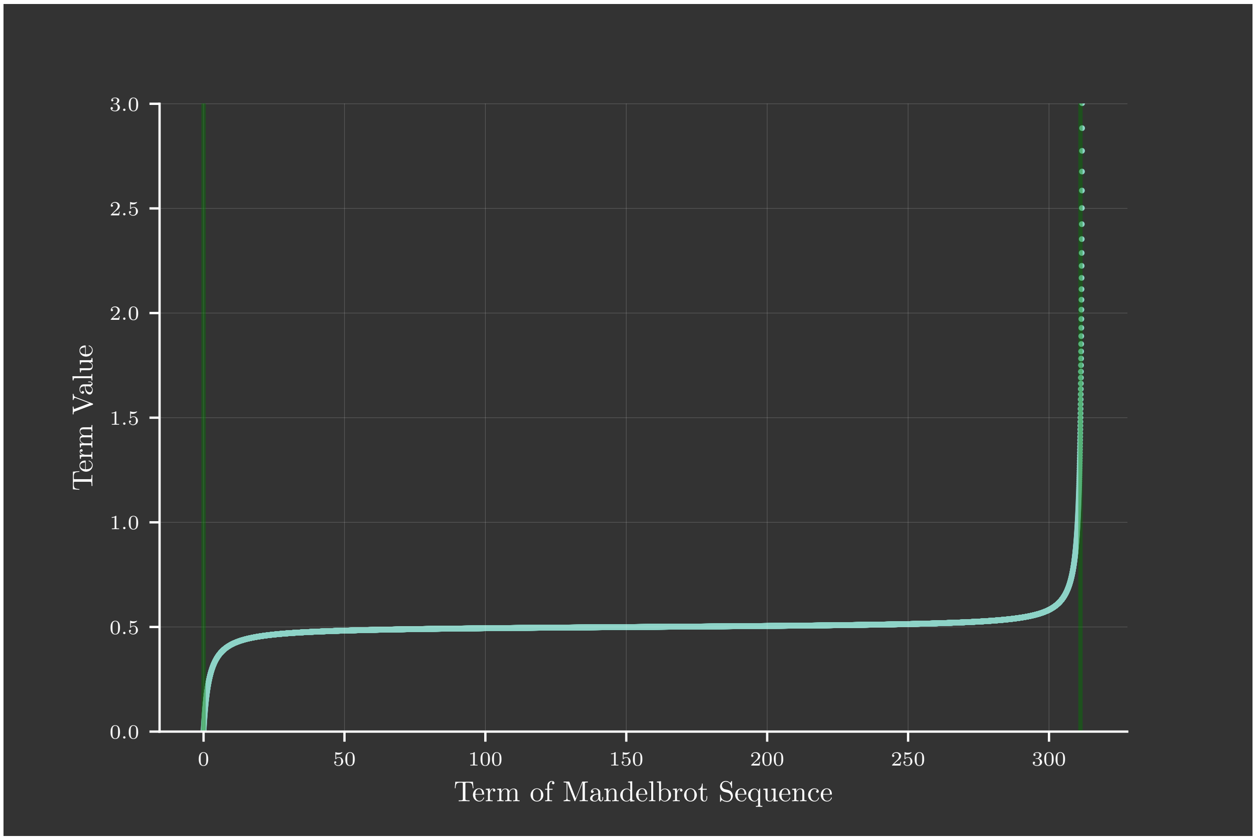


Figure 6: The $\mathcal{M}$ sequence for $c = 0.25 + a$, diverging for $a = 0.0001$

The point at which we cut-off the sequence and consider it to diverge is when the sequence's value is greater than 2, as we know from 2.8. This is only slightly less the the absolute "edge" of the tan graph, which happens to be $\frac{\pi}{\sqrt{a}}$.

Mathematically, this range of terms for when the sequence remains bounded within 2 can be solved for by setting z in the equation for the nth term in (3.6) to 2, and finding n within the range of $0 \leq n \leq \frac{\pi}{\sqrt{a}}$. Solving this gives n, which should approximate π:

Finding when the sequence diverges

$$\sqrt{a}\tan\left(n\sqrt{a} + \arctan\left(\frac{-1}{2\sqrt{a}}\right)\right) + \frac{1}{2} = 2$$

$$\tan\left(n\sqrt{a} + \arctan\left(\frac{-1}{2\sqrt{a}}\right)\right) = \frac{3}{2\sqrt{a}}$$

$$n = \frac{\arctan\left(\frac{3}{2\sqrt{a}}\right) - \arctan\left(\frac{-1}{2\sqrt{a}}\right)}{\sqrt{a}}$$

Using $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$,

$$\arctan\left(\frac{3}{2\sqrt{a}}\right) - \arctan\left(\frac{-1}{2\sqrt{a}}\right) = \arctan\left(\frac{\frac{3}{2\sqrt{a}} + \frac{1}{2\sqrt{a}}}{1 - \frac{3}{4a}}\right)$$

$$n = \frac{\arctan\left(\frac{2\sqrt{a}}{1-\frac{3}{4a}}\right)}{\sqrt{a}} \tag{3.7}$$

But since $-\frac{-\pi}{2\sqrt{a}} \leq \frac{\arctan(x)}{\sqrt{a}} \leq \frac{\pi}{2\sqrt{a}}$, we must shift the range of arctan. To get in the valid range we add $\frac{\pi}{2\sqrt{a}}$ to the inequality, giving $0 \leq \frac{\arctan(x)}{\sqrt{a}} \leq \frac{\pi}{\sqrt{a}}$. It is this boundary of $\frac{\pi}{\sqrt{a}}$ from where the approximation of π comes from. As it will be shown, this n becomes closer and closer to the boundary $\frac{\pi}{\sqrt{a}}$ when smaller values of a are used, explaining the π result.

For smaller and smaller values for $\sqrt{a}$, the ‘stable’ region of the graph becomes longer and longer, meaning that the sequence mostly stays at the same value:

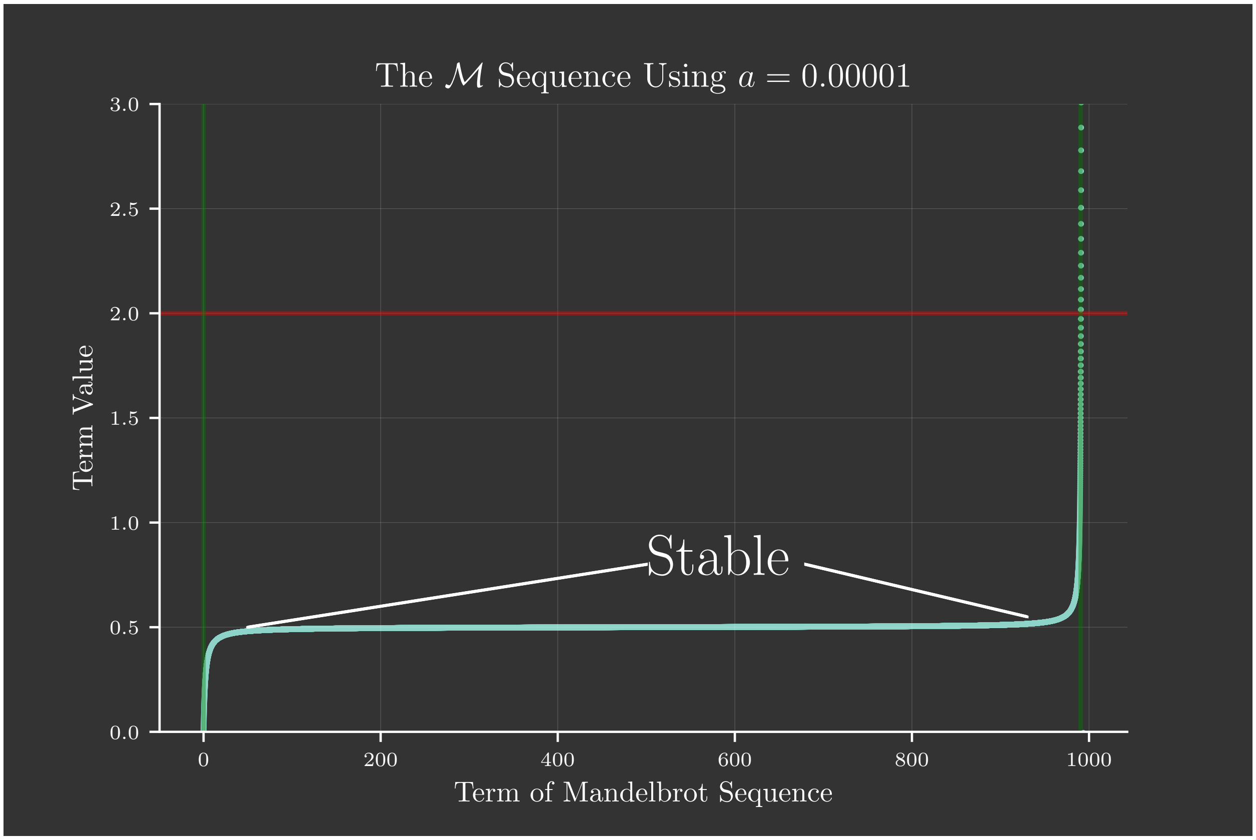


Figure 7: The progression of the $\mathcal{M}$ sequence for a distance of $a = 0.00001$ from the edge. A greater portion of the sequence is now stable and diverges only towards the end.

This means that the number of terms after which the sequence becomes larger than 2 becomes more closer to the absolute boundary of the sequence which is $\frac{\pi}{\sqrt{a}}$. This means that

$$\text{As } a \to 0,\ z_1 \to 0.25^+ \text{ and } n_{div} \to \frac{\pi}{\sqrt{a}} \tag{3.8}$$

Thus, as a tends to 0 and we approach the edge of the Set from 0.25^+, the number of terms to diverge tends to $\frac{\pi}{\sqrt{a}}$.

Thus, n_{div}, the terms to diverge, actually approaches $\frac{\pi}{\sqrt{a}}$ not π directly (hence, the incorrect decimal position in the π approximations) and we can get π by rearranging $n_{div} \to \frac{\pi}{\sqrt{a}}$ in (3.8) to $\sqrt{a} \times n_{div} \to \pi$:

a	$z_1 = 0.25 + a$	n_{div}	$a^{1/2} * n_{div}$
1.0	1.25	2	2.0
0.01	0.26	30	3.0
0.0001	0.2501	312	3.12
0.000001	0.250001	3140	3.14
0.00000001	0.25000001	31414	3.1414
0.0000000001	0.2500000001	314157	3.14157
0.000000000001	0.250000000001	3141591	3.141591

Table 2: Multiplying n_{div} by $\sqrt{a}$ approximates π, as found.

For interest, I tried an $a = 10^{-16}$, this gave a π value correct to 7 d.p., 3.14159263. Granted, this is highly inefficient (took about 5 minutes to calculate).

3.3 Understanding the Stable Section of $\mathcal{M}$

To understand why, as n becomes smaller, the sequence becomes more stable for the middle terms and better approximates π, I studied the sequence through a different form. From (3.2), the equation for the nth term (for approaching 0.25^+) was written in the form:

$$\begin{aligned} z_{n+1} - z_n &= \left(z_n - \frac{1}{2}\right)^2 + a \\ z_{n+1} &= z_n^2 - z_n + \frac{1}{4} + a + z_n \\ z_{n+1} &= z_n^2 + \frac{1}{4} + a \end{aligned} \tag{3.9}$$

Now we have an equation for the next term (using the previous term) made specifically for when the starting term is $z_1 = 0.25 + a$. Using this, we can define a parabola where we replace z_{n+1} with y and z_n with x. Each point on this parabola has an x-value representing the current term and y-value representing the next term.

$$y = x^2 + \frac{1}{4} + a \tag{3.10}$$

Plotting this, we get:

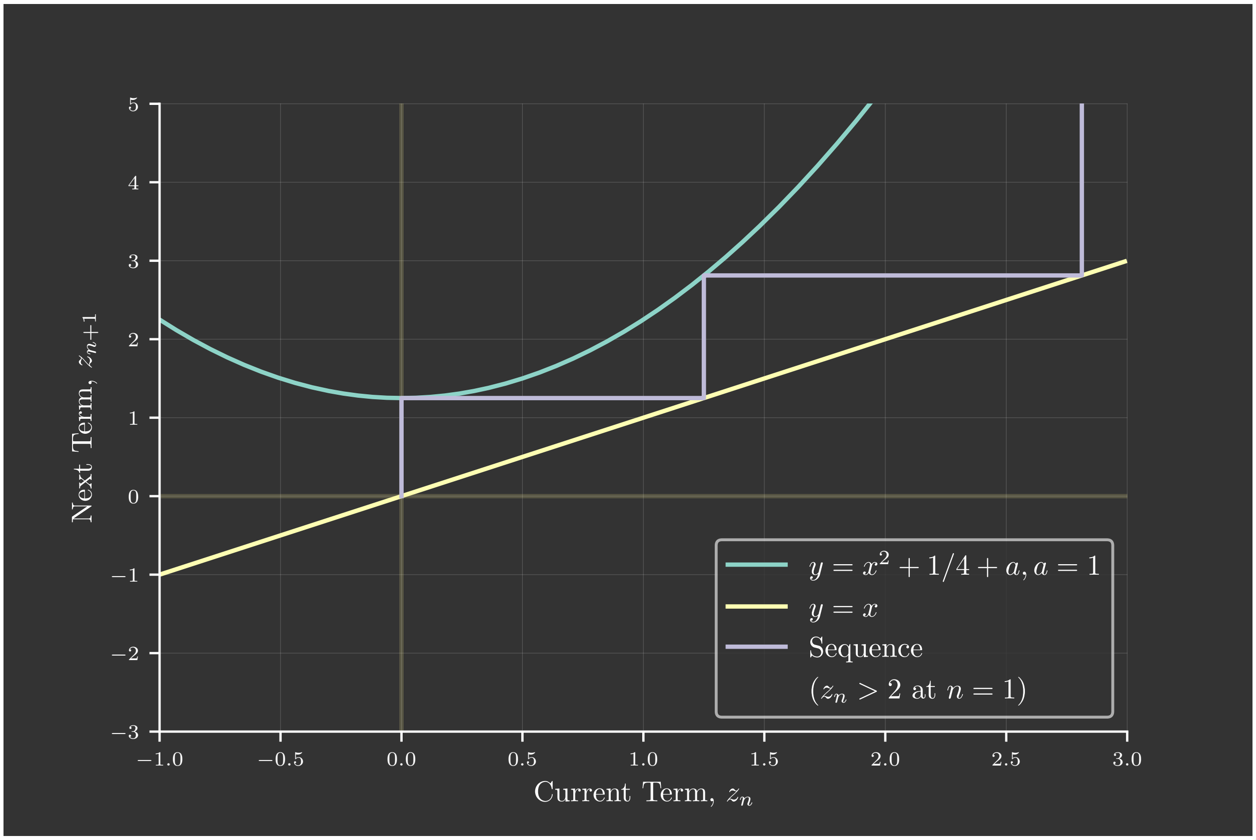


Figure 8: The web diagram for the $\mathcal{M}$ sequence with starting term $z_1 = 0.25 + 1$

Shown in blue is the parabola described. The purple line segments are a form of the $\mathcal{M}$ sequence. On the purple line, the x-value is the current term and y-value is the next term—the line segments can then be thought of as transitions between these values. After a term is computed, this term is inputted back into the parabola to find the term after this, represented as a 'collision' on $y = x$.

Essentially, through this web diagram, we see how the values of the terms change. As seen in Figure 8, for a large a value ($a = 1$), indicating that the starting point is far from the set, the sequence quickly becomes larger than the escape condition of $|z_n| > 2$ (after one iteration). Note: this is why I had chosen the starting term to not have a complex part (only $0.25 + a$)—it allows us to plot the sequence as a parabola, without having to consider an imaginary part.

Here, the a value shifts the parabola up or down, and a smaller a value means there is less space between the parabola and $y = x$, which would increase the intersections with the parabola, explaining the stable behaviour that we saw in the sequence progression in Figure 7, where most of the points remain bounded and only towards the end do the points diverge to infinity.

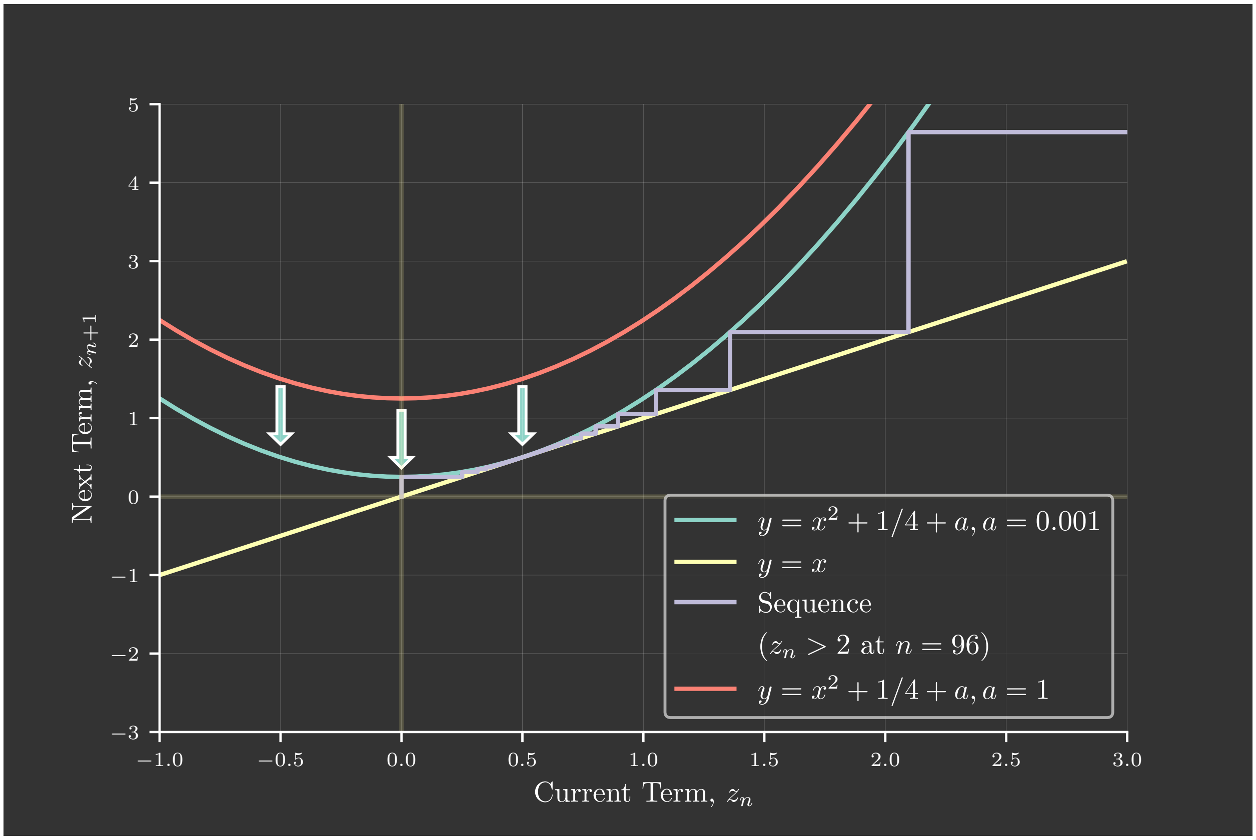


Figure 9: Using a smaller value for a, the parabola shifts closer to $y = x$, indicating the number of iterations before divergence increasing.

As seen above, as $a \to 0$, $y = x$ becomes closer and closer to the parabola, almost becoming a tangent but still keeping some space between, which "allows" for the sequence to still escape to infinity. Interestingly, $y = x$ is tangent to the parabola when $a = 0$, which implies that when $a = 0$ (and the starting term exactly $= 0.25$), the sequence remains bounded and within the Set.

This, then, shows why π is approximated for smaller values of a—the parabola representing the sequence comes closer to $y = x$, making the terms' values stay stable for most of the iterations, and diverge only when closer to the boundary, $\frac{\pi}{\sqrt{a}}$.

4 Further Exploration: Modifying the Equation

The original iterative equation used was:

$$z_{n+1} = z_n^2 + c \tag{4.1}$$

However, I wondered what the Set would look like if this equation were modified, and especially whether the π approximation found for the original set still holds.

For instance, take $z_{n+1} = z_n^3 + c$, now cubing instead of squaring:

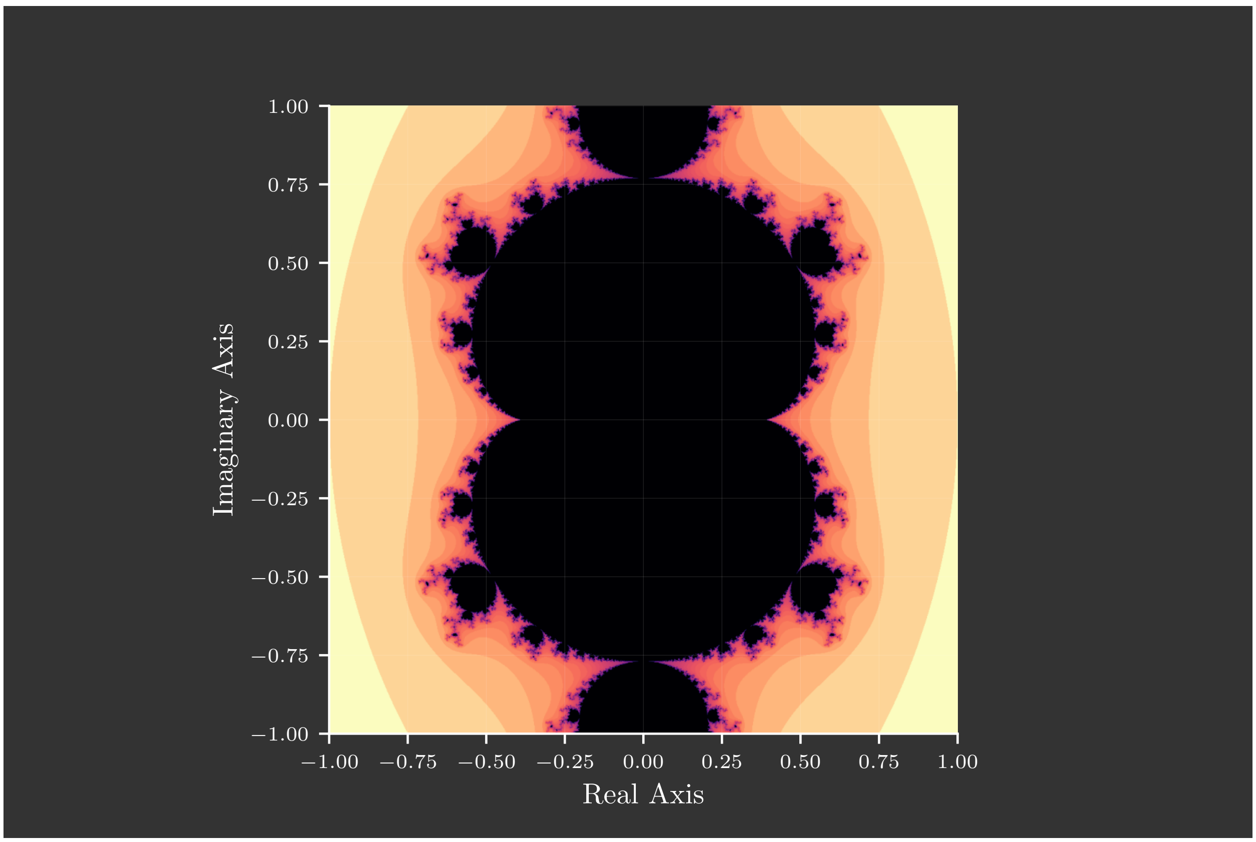


Figure 10: The Mandelbrot Set created with iterative equation $z_{n+1} = z_n^3 + c$ —cubing instead of squaring.

This shape can be explained by the fact that at each iteration, the previous term is now cubed instead of squared, thus, points would now more quickly diverge to infinity. Note that the boundary condition of $|z_n| > 2$ still holds [2].

[2] The only difference is that the factor the current term is multiplied by to get the next term is now greater as it is being now raised to the 3rd power instead of the second power

This time, the 'edge' of the set was not as clear (as opposed to the point 0.25 being known as the edge for the original set). Thus, I used the web diagram to see at which point the equation $z_{n+1} = z_n^3 + c$—in cubic form ($y = x^3 + c$)—has a tangent to $y = x$ (as this means the sequence does not escape, as seen in Figure 9).

We can formulate this problem into the following, where b represents the 'edge' to find:

Finding the edge of the set

Find b in $f(x) = x^3 + b$ such that $f'(p) = 1$ at p

and that $f(p) = x$ (tangent to $y = x$) for some x-value, p

$$f'(x) = 3x^2 = 1 \text{ (tangent to } y = x)$$

$$\implies 3p^2 - 1 = 0$$

$$p = \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$$

We can eliminate $p = -\frac{1}{\sqrt{3}}$ as we are interested in $p > 0$

Also,

$$f(x) = x^3 + b = x \text{ (must meet tangent at } x = p)$$

$$\implies p^3 + b = p$$

$$b = \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} \tag{4.2}$$

This gives the following point as the 'edge':

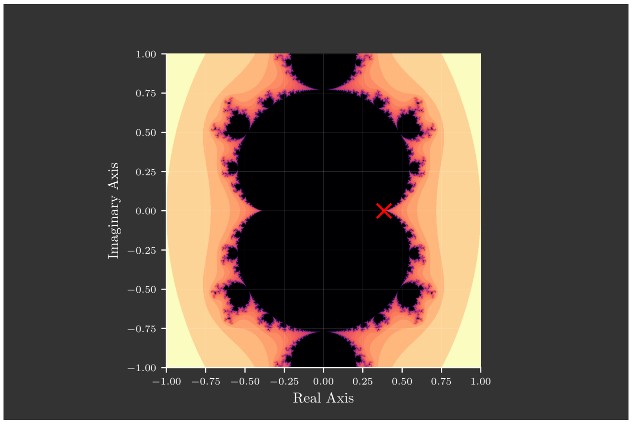


Figure 11: The edge of the cubed Mandelbrot set (the red cross) to be approached.

However, when approaching this point, $(b, 0)$, using the same methodology as previously, we find that the list of values now converge as such:

a	$z_1 = b + a$	n_{div}
1.0	1.38490017945975060	2
0.01	0.39490017945975060	23
0.0001	0.38500017945975057	238
0.000001	0.38490117945975055	2386
0.00000001	0.38490018945975060	23870
0.0000000001	0.38490017955975060	238708
0.000000000001	0.38490017946075056	2387033

Table 3: Approaching the edge of the Mandelbrot set with an iterative equation of $z_{n+1} = z_n^3 + c$.

The number of terms before the sequence diverges, n_{div}, no long approximates π. This likely results from the function of the nth term now not containing a trigonometric function, as it previously did, containing arctan to relate the sequence to π. The differential for the sequence would look like the following:

$$\begin{aligned} z_{n+1} &= z_n^3 + \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} + a \\ \frac{dz}{dn} &= z_n^3 - z_n + \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} + a \end{aligned} \tag{4.3}$$

The integral of this to find the nth term could not be found using standard or computational approaches (see Appendix 6.3). However, we still see he n_{div} converging to some value, though the sequence may simply be diverging at an arbitrary value.

Overall, it seems that the π approximation only holds for the $z_{n+1} = z_n^2 + c$ sequence.

5 Conclusion and Discussion

Thus, I have found how π originates from the Mandelbrot Set, though this seems to only be the case for the power-two iterative sequence ($z_{n+1} = z_n^2 + c$). The methodology was limited in that the change in term value was approximated as a continuous differential (though the term numbers were discrete) and the limited precision of variables during computing, which especially mattered while using powers of up to -14.

Future investigations could involve modifying other aspects of the iterative equation, for instance the starting constant, z_0. Additionally, other points on the Mandelbrot Set could be approached, such as $(-0.75, a)$, where the edge of the Set is now approached from the imaginary axis.

6 Bibliography

1. Abbas, Mujahid, et al. "Generation of Julia and Mandelbrot Sets via Fixed Points." Symmetry, vol. 12, no. 1, 2020, p. 86., https://doi.org/10.3390/sym12010086.
2. (n.d.). Integral Calculator. https://www.integral-calculator.com
3. Klebanoff, A. (2001). π IN THE MANDELBROT SET. World Scientific. https://www.doc.ic.ac.uk/ jb/teaching/jmc/pi-in-mandelbrot.pdf
4. Milnor, John W. "Dynamics in One Complex Variable: Introductory Lectures." ArXiv.org, 20 Apr. 1990, https://doi.org/10.48550/arXiv.math/9201272.
5. Web diagram. (2003, October 28). Wolfram MathWorld Resource. https://mathworld.wolfram.com/WebDiagram.html

7 Appendix

7.1 Program: Plotting the Set

```
import numpy as np
from numba import jit
# from PIL import Image, ImageDraw
import ipywidgets as widgets
# from ipywidgets import HBox, VBox
# import numpy as np
from decimal import Decimal
import matplotlib.pyplot as plt
import pandas as pd
import matplotlib
import math
#%matplotlib tk ENABLE FOR LIVE EXPLORER <----#######
from matplotlib.backend_bases import MouseButton
# matplotlib.use('TkAgg')
# matplotlib.rcdefaults()
```

```
plt.style.use('dark_background')



#for mouse clicks
# from IPython.display import display
# %matplotlib inline

@jit
def mandel_iters(c_re, c_im, pow_seq, z_re = 0, z_im = 0, max_iter = 100):
    iter = 0
    c = complex(c_re, c_im)
    z = complex(z_re, z_im)
    while abs(z) < 2 and iter < max_iter:
        z=z**pow_seq+c
        iter += 1

    return math.log(iter)
def mandel_iters_cl(c_re, c_im, pow_seq, z_re = 0, z_im = 0, max_iter = 100):
    iter = 0
    c = complex(Decimal(c_re), Decimal(c_im))
    z = complex(Decimal(z_re), Decimal(z_im))
    while abs(z) < 2 and iter < max_iter:
        z=z**pow_seq+c
        iter += 1

    return iter

@jit
def julia_iters(z_re, z_im, c_re = 0, c_im = 0, max_iter = 100):
    iter = 0
    while z_re*z_re + z_im*z_im < 4 and iter < max_iter:
        z_re_temp = z_re*z_re - z_im*z_im + c_re
        z_im = 2*z_re*z_im + c_im
        z_re = z_re_temp

        iter += 1
```

```
    return math.log(iter)

def map(num, num_low, num_high, to_low, to_high):
    return (num-num_low)/(num_high-num_low) * (to_high-to_low) + to_low

def plot_mandel(x_start, x_end, y_start, y_end, n_points, max_iters):
    x_points = int((x_end-x_start)*n_points/(y_end-y_start))
    plot = np.zeros((n_points, x_points))

    for x_index, x in enumerate(np.linspace(x_start, x_end, x_points)):
        for y_index, y in enumerate(np.linspace(y_start, y_end, n_points)):
            iters = mandel_iters(x, y, max_iter=max_iters)
            plot[y_index][x_index] = (0 if iters == max_iters else iters)

    plt.figure(figsize=(10, 10))
    plt.imshow(plot, cmap='turbo', interpolation='spline16')

def mandel_iters_c(c_re, c_im, z_re = 0, z_im = 0, max_iter = 100):
    iter = 0
    while z_re*z_re + z_im*z_im < 4 and iter < max_iter:

        z_re_temp = z_re*z_re - z_im*z_im + c_re
        z_im = 2*z_re*z_im + c_im
        z_re = z_re_temp

        iter += 1

    return math.log(iter)
    #Return logged iters for smoothened curve (reduce large values
    , so gradient smoother)
```

7.2 Program: The Web Diagram

```
#Parabola converge
matplotlib.rcParams.update({"text.usetex": True})
```

```
%matplotlib inline
a=1
og_x=0
og_y=0
global z_current
def p(z, a):
    return z**2+1/4+a
def ref(x, y):
    return (y, x)

def web(initial=0, iters=10):
    current = initial
    out = pd.DataFrame(columns=['prev', 'current'])
    iters_t=0
    for i in range(0, iters, 2):
        try:
            out.loc[i]=current, p(current, a)
            current=p(current, a)
            out.loc[i+1]=current, current
            if current < 2:
                iters_t+=1
        except:
            break
    out.loc[-1]=0,0
    out.index = out.index + 1
    out = out.sort_index()
    return out, iters_t

steps, t_iters = web(0, 1000)
n = np.arange(-1, 3, 0.01)
plt.plot(n, [p(n, a) for n in n], label = f'$y=x^2+1/4+a, a = {a}$', alpha=1)
plt.plot(n, n, alpha=1, label='$y=x$')
plt.ylim(-3,5) # can set to lim of ...
plt.xlim(-1, 3)
plt.xlabel('Current Term, $z_n$')
plt.ylabel('Next Term, $z_{n+1}$')
```

```
plt.axvline(0, c='khaki', alpha=0.2)
plt.axhline(0, c='khaki', alpha=0.2)
plt.plot(steps['prev'], steps['current'], label='Sequence') #, c='lightgreen'
plt.grid(alpha=0.2)
plt.plot([],[],label=f'($z_n>2$ at $n={t_iters}$)', c='none')
l = plt.legend()
plt.savefig(f'zn_parabola_{a}.pdf')
```

7.3 Plotting the Function for the $\mathcal{M}$ Sequence

```
#Plot tan sequence num
%matplotlib inline
a=0.0001
def z(n):
    return np.sqrt(a)*np.tan(n*np.sqrt(a)+np.arctan(-1/(2*np.sqrt(a))))+1/2

n = np.arange(-0.004/a, (np.pi/np.sqrt(a))+0.004/a, 0.05)
plt.scatter(n, [z(n) for n in n], s=1)
plt.ylim(-10, 10)
plt.xlabel('Term of Mandelbrot Sequence')
plt.ylabel('Term Value')
plt.axvline(0, color='green', alpha=0.4)
plt.axvline((np.pi/np.sqrt(a))-3, color='green', alpha=0.4)
plt.grid(alpha=0.2)
plt.savefig(f'zn_{a}.pdf')
```

7.4 nth Term for the $\mathcal{M}$ Cubed Sequence

The antiderivative computed was partially solved; however, it was overall unknown as there was no polynomial found that would fit the form:

$$\left(\sum_{\left\{ Z:\, 3^{\frac{3}{2}} a + 3^{\frac{3}{2}} Z^3 - 3^{\frac{3}{2}} Z + 2 = 0 \right\}} \frac{3^{\frac{3}{2}} \ln\left(|x - Z|\right)}{3^{\frac{5}{2}} Z^2 - 3^{\frac{3}{2}}} \right) + C$$